Deduction and construction of a discrete probability function (Binomial and Geometric) with analytical development of expected value and variance

Deducción y construcción de una función de probabilidad discreta (Binomial y Geométrica) con desarrollo analítico de valor esperado y varianza

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Abstract

The present work deals with the construction and deduction of a discrete probability function, where the objective is the understanding of the mathematical foundations of probability for its correct application, interpretation and verification in solving probability problems. For this purpose we rely on the theory of didactic situations of Brousseau (1997) and Sadovsky (2005). Teaching strategies are proposed with didactic model materials that improve student learning by promoting the analysis and understanding of various foundations of probability (Panizza, 2003). The importance of close communication between teachers and students in the teaching-learning process for the resolution, interpretation and verification of probability problems is highlighted. The software used in this research was Wolfram Mathematica, which facilitates mathematical writing, calculations, and the construction of graphs using a simple interface. This software makes work easier by fostering student autonomy in the development of skills for analysis and problem solving. We believe that these materials will contribute to the improvement of teaching-learning processes on the fundamentals and concepts of probability.

Discrete probability function, Expected value, Higher education

Resumen

El presente trabajo trata la construcción y deducción de una función de probabilidad discreta, donde el objetivo es la comprensión de los fundamentos matemáticos de la probabilidad para su correcta aplicación, interpretación y comprobación en la resolución de problemas probabilidad. Para este fin nos apoyamos en la teoría de situaciones didácticas de Brousseau (1997) y Sadovsky (2005). Se proponen estrategias de enseñanza con materiales de modelos didácticos que mejoren el aprendizaje del estudiante fomentando el análisis y la comprensión de diversos fundamentos de la probabilidad (Panizza, 2003). Se destaca la importancia de la estrecha comunicación entre profesores y estudiantes en el proceso la enseñanza-aprendizaje para la resolución, interpretación y comprobación de problemas de probabilidad. El software utilizado en esta investigación fue Wolfram Mathematica, mismo que facilita la escritura matemática, los cálculos y la construcción de gráficas utilizando una interfaz sencilla. Este software facilita el trabajo fomentando la autonomía del estudiante en el desarrollo de las habilidades para el análisis y la resolución de problemas. Consideramos que estos materiales permitirán contribuir en la mejora de los procesos de enseñanza-aprendizaje sobre los fundamentos y conceptos de probabilidad.

Función de probabilidad discreta, Valor esperado, Educación superior

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Introduction

In order to build and deduce a discrete probability function, in the binomial case we start from an ancient activity that continues to be carried out today, such as sowing seeds. At the same time, we use definitions and fundamental concepts in order to develop analytically the demonstrations of expected value and variance, through the application of algebraic procedures. This didactic strategy allows the construction and understanding, in a natural, inductive and deductive way, of the concepts involved. The teaching of probability and statistics has presented a great development in recent years due to its increasing application in various fields of science, technology and social and administrative disciplines.

Many institutions and educational centers in the world have dedicated great efforts to design and update curricula and specific materials, to alleviate the difficulties in the teaching-learning process of probability and improve the quality of education.

For this work we initially rely on the Theory of Didactic Situations due to the conceptual nature of probability and its application (Brousseau, 2002) (Brousseau, 1986) (Sadovsky, 2003) (Barreiro and Casseta, 2015) (Pérez y Beltrán, 2011) (Pérez y Pérez, 2018). The theory of didactic situations makes it possible to generate adequate conditions for the analysis of mathematical knowledge, under the hypothesis that they are not built spontaneously (Panizza, 2003). That is, it is based on the premise that knowledge is not transmitted from one person to another automatically, but rather that the individual constructs his or her own knowledge. In this sense, the role of the teacher is very important because a rigorous design and a judicious and appropriate choice of problems depend on him.

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The second approach is the one referring to the constructivist theory that allows developing heuristic strategies so that students choose appropriate methods for solving problems when facing real world situations. This didactic approach stimulates curiosity in students to understand the problem posed. The student thus develops the concepts and procedures to understand and solve problems. Furthermore, students are involved in a permanent process to acquire knowledge in a real context, since, for constructivism, the most important thing is not the new knowledge itself, but to acquire new skills, which allows students to apply what is already known to the understanding of a new problem.

Justification

There are several reasons that justify this work: the first reason is related to the teaching work: it is known that mathematical issues and therefore probability are complex for students of all educational levels, as reported by the literature on Statistical Education (Batanero and Godino, 1997).

The second reason refers to the importance of probability in school curricula in the educational system in Mexico. For example, at the basic level, students finish their instruction without having assimilated the knowledge and understanding of meanings related to probability.

This partly has to do with the fact that certain topics are often not covered, even if they are within the curricular guidelines, and when they are covered, it is done procedurally and not conceptually. The third reason is that, just as students have difficulties understanding probability concepts, teachers have great difficulty in teaching in a comprehensive and clear way.

Problem Statement

According to the teaching experience and research reports presented in different national and international forums on Statistics and probability, we can affirm that the didactic resources to support teaching are scarce. On the other hand, the textbooks used to teach probability and statistics give more importance to procedure than to understanding; furthermore, the exploratory approach is reduced.

On the other hand, some textbooks such as the following: William Mendenhall (2013), (Mood, Graybill & Boes, 1974); Ronald E. Walpole (2012) only mentions the functions and their respective parameters, but omits detailed demonstrations, and even more so, in the case of George C. Canavos (1988) presents some incomplete developments.

Another important aspect is the scarce national bibliography of probability texts, since they are mainly published in their original language (English, French, German, Russian, Italian, etc.). Books in other languages come from diverse cultural contexts and are focused on the educational systems from which they come. Consequently, these books do not take into consideration the cultural identity of Mexico and its social and educational context. Due to the above, it is considered important to have our textbooks appropriate to the social and educational context of Mexico.

Hypothesis

The selection of didactic strategies that use appropriate examples to promote interest and active participation of students in understanding and solving probability mathematical problems will promote self-learning and the development of skills to understand more complex topics in the classroom.

General objectives

Design didactic strategies for the construction and formulation of analytical demonstrations of the discrete probability function, expected value and variance, through procedures and use of algebraic artifices, identities and differential calculus.

Specific objectives

To have didactic materials that allow the student to strengthen their autonomy and develop their skills and abilities at a higher level. Among them the following can be mentioned:

- Express ideas clearly.
- Structure ideas logically.
- Structure graphs, tables and diagrams that help to obtain the desired result.
- Use appropriate language and mathematical representation.
- Selection of appropriate mathematical tools.
- Demonstrate knowledge and understanding.
- Apply mathematics in different media and contexts.
- Apply problem solving techniques.

Theoretical Frame

Many math problems require proofs. This challenge is a valuable opportunity for the student to apply previous knowledge and develop skills through problem solving. It can be affirmed that the development of these skills is the result of the effort and personal work of each student throughout the teaching-learning process. It is difficult for a student to become a good mathematical problem solver just by passively reading a book. However, the effort on the part of the student to understand and understand a mathematical problem, as well as the use of appropriate techniques can be of great help to solve the problem.

It is common that, in various mathematical sub-disciplines, it is required to understand key concepts and the relationships between them to solve various mathematical problems.

It is also known that problem solving activities promote the development of skills and abilities in the student. This fact is an argument that justifies the importance of including this approach within the study plans. At the same time, this approach is considered essential in the educational process, since it allows students to understand the scope and usefulness of mathematics in the real world. Throughout life, every individual will encounter problems that must be faced, understood and resolved both inside and outside of school. Therefore, it is considered relevant in developing in the student the ability to solve problems in the classroom.

In a lecture delivered in 1968 George Polya said: “It is well justified that all mathematics texts contain problems. The problems can even be considered as the most essential part of mathematics education” (Pérez and Beltrán, 2011) (Polya, 1965).

Guzmán (1984) comments that “what we should above all provide our students through mathematics is the possibility of acquiring adequate thought habits for solving mathematical and non-mathematical problems. What good can a few theorems and properties related to entities with little meaning serve in your mind if you are going to leave them there hermetically locked up? Problem solving has rightly been called the heart of mathematics, because that is where you can acquire the true value that has brought and attracts mathematicians of all ages. Facing appropriate problematic situations is where motivations, attitudes, habits, ideas for the development of tools can result, being the life of mathematics. When solving problems, you fundamentally learn to understand how our own reasoning works, to master our moods and to increase our self-confidence”.

Methods

Examples of probability problems were selected to stimulate the student's interest in analysis and reflection (deep observation of the elements involved and their relationships between them).

From this reading, the reformulation of the problem is proposed to give way to an algebraic development, using it as a guide in the search for ideas for the solution of the problem. In general, this technique leads us at the same time to develop a mathematical process through algebraic tricks that allow a new structuring of the problem and improve its understanding until the desired result is obtained. In short, this is a process of analysis and synthesis.

This didactic strategy has an initial empirical approach in order to solve real world problems. These problems require quantity quantifications and at the same time, they are associated with a qualitative analysis.

Campistrous P. L. and Rizo C. (1996), with great objectivity, point out that traditional procedures are aimed at the actions carried out by teachers in the teaching process where the importance of active student participation is not highlighted. Considering the above, these authors express the following aspects that characterize traditional teaching:

- The stimulation is indirect
- The forms of performance generalized in the student are not achieved, even when they are necessary for life.
- The problems are focused on developing calculus skills and not as a teaching object.
- The difficulty of the problems and their parameters are not very precise, which confuses or makes it impossible to build analogies.
- Particularly, in mathematical analysis problems, the meanings are not adequately worked on.
- Likewise, they present a series of techniques that allow them to face the challenges to solve more complex problems.

Kind of investigation

We can locate this Research in descriptive, characterized by the depth of the mathematical analysis and the development of techniques and strategies to pose and solve problems where the interpretation and verification acquire relevance in its application.

Theoretical Methods

In these sample problems we address algebraic procedures by applying mathematical artifices to develop proofs of expected value and variance. To process the examples, the Wolfram Mathematica software was used, facilitating the preparation of graphs and calculations. The previous knowledge that is required is knowledge of algebra and differential calculus of higher level.

Development Methodology

Definitions and basic concepts of a discrete random variable:

Expected Value

$$E(X) = \sum_{x=1}^{N} x f_x(x) \quad (1)$$

$$Var(X) = \sum_{x=1}^{N} (X - E(X))^2 f(X) \quad (2)$$

$$Var(X) = E(X^2) - [E(X)]^2 \quad (3)$$

Next, the deduction and construction of a probability function is presented, as well as the analytical demonstrations of expected value and variance respectively, resorting to the application of a meticulous algebraic work supported by artifices and indirect expressions of derivatives and expected value.

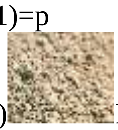
Results

The deduction and construction of the Binomial and Geometric probability functions are presented together with the definitions and proofs of expected value and variance below.

Deduction and construction of the Binomial Probability Distribution Function

4 corn seeds are planted under the same and independent conditions, where the variable X represents the number of seeds that will germinate, the following table shows the possible cases with their respective probabilities.

Coding: 1= success  P(1)=p

0=failure (does not germinate)  P(0)=q

Éxitos X	 1 | 2 | 3 | 4	Probabilidad $f(x) = \binom{4}{x} p^x q^{4-x}$
4	1 | 1 | 1 | 1	$P(4) = (1)\, p^4 q^{4-4}$
3	1 | 1 | 1 | 0 1 | 1 | 0 | 1 1 | 0 | 1 | 1 0 | 1 | 1 | 1	$P(3) = (4)\, p^3 q^{4-3}$
2	1 | 1 | 0 | 0 1 | 0 | 1 | 0 1 | 0 | 0 | 1 0 | 1 | 1 | 0 0 | 1 | 0 | 1 0 | 0 | 1 | 1	$P(2) = (6)\, p^2 q^{4-2}$
1	1 | 0 | 0 | 0 0 | 1 | 0 | 0 0 | 0 | 1 | 0 0 | 0 | 0 | 1	$P(1) = (4)\, p^1 q^{4-1}$
0	0 | 0 | 0 | 0	$P(0) = (1)\, p^0 q^{4-0}$

Table 1 Generalizing the problem for n seeds

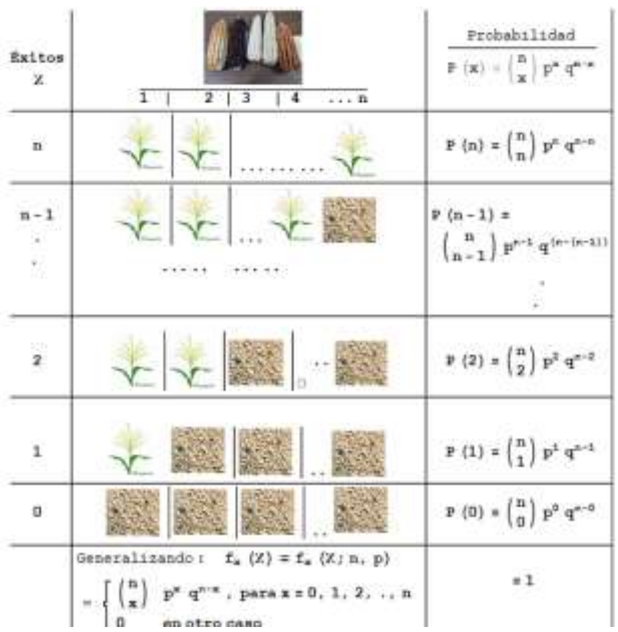


Table 2

Expected value: $E(X) = \sum_{x=0}^{n} x\binom{n}{x} p^x q^{n-x}$

$= 0\binom{n}{0}p^0q^{n-0} + 1\binom{n}{1}p^1q^{n-1} + \ldots n\binom{n}{n}p^nq^{n-n}$

$$\sum_{x=1}^{n} x \frac{n(n-1)!}{x(x-1)!(n-x)!} p^x q^{n-x} p^1 p^{-1} = np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!(n-x)!} p^{x-1} q^{n-x}$$

variable change $y = x - 1 ⇒ x = y + 1$

$x \to n \qquad y = n - 1$

$x \to 1 \qquad y = 0$

$$np \sum_{y=0}^{n-1} \frac{(n-1)!}{y!(n-1-y)!} p^y q^{n-1-y} = np \sum_{y=0}^{n-1} \binom{n-1}{y} p^y q^{n-1-y}$$

$$= np \sum_{y=0}^{n-1} \binom{n-1}{y} p^y q^{n-1-y}$$

By Newton′s Binomial

$$(x+y)^n=\sum_{k=0}^{n}\binom{n}{k}y^k x^{n-k}$$

$$(p+q)^{n-1}=\sum_{y=0}^{n-1}\binom{n-1}{y}p^y q^{n-1-y}=1$$

$\therefore E(X)=np$

Variance: $Var(X)=E(X^2)-[E(X)]^2$

$$E[X(X-1)]=\sum_{x=0}^{n}[X(X-1)]f(x)$$

$$=\sum_{x=0}^{n}[X(X-1)]\binom{n}{x}p^x q^{n-x}$$

$$=0\binom{n}{0}p^0q^{n-0}+1(1-1)\binom{n}{1}p^1q^{n-1}+2(2-1)\binom{n}{2}p^2q^{n-2}\ldots+n(n-1)\binom{n}{n}p^nq^{n-n}$$

$$=0+0+\sum_{x=2}^{n}[X(X-1)]\binom{n}{x}p^xq^{n-x}$$

$$=0\binom{n}{0}p^0q^{n-0}+1(1-1)\binom{n}{1}p^1q^{n-1}+\sum_{x=2}^{n}[X(X-1)]\binom{n}{x}p^xq^{n-x}$$

$$=0\binom{n}{0}p^0q^{n-0}+1(1-1)\binom{n}{1}p^1q^{n-1}+\sum_{x=2}^{n}[X(X-1)]\binom{n}{x}p^xq^{n-x}$$

$$=\sum_{x=2}^{n}[X(X\ -1)]\frac{n(n-1)!}{X(X-1)(X-2)!(n-X)!}p^xq^{n-x}$$

Variable change $y=x-2,\quad x=y+2$

$x=n \qquad y=n-2$

$x=2 \qquad y=0$

$$=n(n-1)\sum_{x=2}^{n}\frac{(n-2)!}{(x-2)!(n-x)!}p^xp^2p^{-2}q^{n-x}=$$

$$p^2n(n-1)\sum_{x=2}^{n}\frac{(n-2)!}{(x-2)!(n-x)!}p^yq^{n-2-y}$$

By Newton′s Binomial

$$(x+y)^n=\sum_{k=0}^{n}\binom{n}{k}y^kx^{n-k}$$

$$\therefore\sum_{y=0}^{n-2}\frac{(n-2)!}{(y)!(n-2-y))!}p^yq^{n-2-y}=(p+q)^{n-2}$$

$$\sum_{y=0}^{n-2}\binom{n-2}{y}q^yp^{n-2-y}$$

$$p+q=1\therefore(p+q)^{n-2}=\sum_{y=0}^{n-2}\binom{n-2}{y}p^yq^{n-2-y}$$

$E[X(X-1)]=E[X^2]-E[X]=p^2n(n-1)$

E[X] = np

$E[X^2]=p^2n(n-1)+np\therefore E[X^2]$

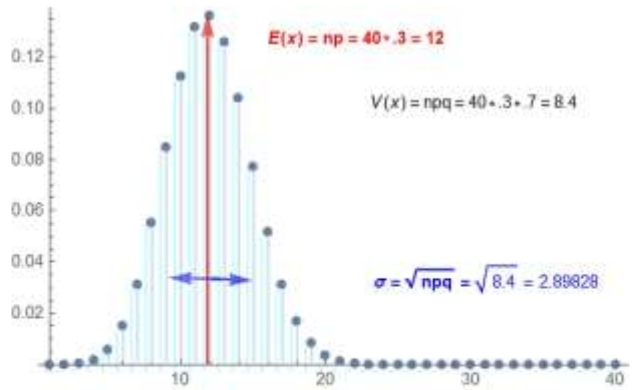
$=p^2n(n-1)+np=p^2n^2-p^2n+np$

$V\,ar(X)=p^2n^2-p^2n+np-(np)^2$

$=np(1-p)=\mathbf{npq}$

DiscretePlot[

Evaluate@Table[PDF[BinomialDistribution[40, p], k], {p, {0.3}}], {k, 40}, PlotRange -> All, PlotMarkers -> Automatic]



Graph 1

Deduction and construction of the Geometric Probability Distribution Function

We start from a practical application problem: A real estate company sells properties, it is known that a property is sold with a probability V on average. It is known that the sale on one day is independent of the sale on any other day; Find the probability distribution for X, the number of days that there is no sale between two sales

.Sell =**V** Do not sell=**N** $P(V)=V$ $P(N)=N$

$V+N=1\ y\ (1-V)=N,\ 0<N<1$

What can happen? Helps define events. In how many ways can each outcome occur? With the help of tables, diagrams, drawings, colors and counting techniques, it will surely be easier to assign probabilities to each result attached to the axioms and theorems of probability.

X → / i	0	1	2	.	n
0	VV				
1	NVV	VNV			
2	NNVV	NVNV	VNNV		
3	NNNVV	NNVNV	NVNNV	.	
.	.	.	.	.	
n	NN . . NVV	NN . . NVNV	NN . . NVNNV	.	VN…NV

Table 3

Assigning probabilities to each event requires having clear concepts and applying the fundamentals of probability.

Next, the table shows the cases of the number of days of no sale that there are between two days that there are sales

Filas	X→ / n↓	X = 0	X = 1	X = 2	..X = n-1	X = n
1	0	VV				
2	1	NVV	VNV			
3	2	NNVV	NVNV	VNNV		
4	3	NNNVV	NNVNV	NVNNV	...	
. k - 1	. n-1	. .	. .	. .	. N . . NVN . . NV	
k	n	NN . . NVV	NN . . NVNV	N . NVNNV	. N . . NVN . . NV	VN...NV
	P (x)	$V^2 \sum_{i=0}^{n} N^i$	$V^2 \sum_{i=1}^{n} N^i$	$V^2 \sum_{i=2}^{n} N^i$	$.V^2 \sum_{i=n-1}^{n} N^i$	$V^2 N^n$

Table 4

Notice that in each row of the table there are k non-empty cells, and each cell in the row has two V and (k-1) N letters.

The last row represents the sums of the probabilities of each column. It is worth mentioning that these sums can be rewritten as:

$$\mathbf{V^2 \sum_{i=0}^{n} N^i} \quad (4.1)$$

$$\mathbf{V^2 \sum_{i=1}^{n} N^i = V^2 \sum_{i=0}^{n} N^i - V^2} \quad (4.2)$$

$$\boldsymbol{V^2 \sum_{i=2}^{n} N^i = V^2 \sum_{i=0}^{n} N^i - V^2 - N\, V^2 = V^2 \sum_{i=1}^{n} N^i - V^2 N} \quad (4.3)$$

$$\boldsymbol{V^2 N^n = V^2 \sum_{i=0}^{n} N^i - V^2 - N\, V^2 - N^2 V^2 - \ldots - N^{n-1} V^2 = V^2 \sum_{i=n-1}^{n} N^i - V^2 N^{n-1}} \quad (4.4)$$

Fila	X = 0	X = 1	X = 2	..	X = n
1	VV				
2	NVV	VNV			
3	NNVV	NVNV	VNNV		
.	.	.	.	..	
k	N ⋯ N VV (k-1)	N ⋯ N V N V (k-2) (+1)	.	..	V N⋯N V (k-1)

Table 5

Solution: Developing the first sum (Table 4) and substituting probability values, respectively: $P(X=0) = V^2 \sum_{i=0}^{n} N^i = V^2 \sum_{i=0}^{n} N^i = V^2\{N^0 + N^1 + N^2 + \cdots N^{n-1} + N^n\}$ (5)

$$S = \sum_{i=0}^{n} N^i = \{N^0 + N^1 + N^2 + .. N^{n-1} + N^n\} = \{1 + N^1 + N^2 + .. N^{n-1} + N^n\} \quad (6)$$

We multiply (6) by N, we obtain:

$$NS = (N)\{1 + N^1 + N^2 + .. N^{n-1} + N^n\} = \{N^1 + N^2 + N^3 + .. N^n + N^{n+1}\} \quad \mathbf{(7)}$$

Resetting (6) - (7) we get:

$$S = \{1 + N^1 + N^2 + N^3 \ldots.. + N^{n-1} + N^n\}$$
$$-NS = -\{N^1 + N^2 + N^3 + \ldots. + N^{n-1} + N^n + N^{n+1}\}$$
$$= 1 - N^{n+1}$$

$$S - NS = 1 - N^{n+1} \Rightarrow S = \frac{1 - N^{n+1}}{1 - N}, n \to \infty \therefore \lim_{n \to \infty} N^{n+1} = 0$$

$$S = \frac{1}{1-N} = \frac{1}{1-N} = \frac{1}{V} \therefore \lim_{n \to \infty} V^2 \sum_{i=0}^{n} N^i \quad (8)$$

$$\lim_{n \to \infty} V^2 \sum_{i=0}^{n} N^i = V^2 S = V^2\left(\frac{1}{V}\right) = V$$

$$\therefore \mathbf{P(X = 0) = V}$$

Now we can calculate the probabilities for each value of X, according to the last row of table 4.

P (X = 0)	$V^2 \sum_{i=0}^{n} N^i$ $= V N^0$
P (X = 1)	$V^2 \sum_{i=0}^{n} N^i - V^2 =$ $= V - (V^2) = V(1 - V) = V N^1$
P (X = 2)	$V^2 \sum_{i=1}^{n} N^i - V^2 N =$ $V(N) - (V^2)(N) = V(N)(1 - V) = V N^2$
P (X = 3)	$V^2 \sum_{i=2}^{n} N^i - V^2 N^2 =$ $V(N^2) - (V^2)(N^2) = V(N^2)(1 - V) = V N^3$
... P (X = n - 1)	.... $V N^{n-1}$
P (X = n)	$V^2 \sum_{i=n-1}^{n} N^i - V^2 N^{n-1} =$ $V(N^{n-1}) - (V^2)(N^{n-1}) = V(N^{n-1})(1 - V) = V N^n$

Table 6

We can deduce by induction the probability function: $P(X) = VN^i$ which must fulfill:

The sum of all probabilities must be equal to one (that is, the limit when n→∞ of the sum of the last row of table 2 must be equal to one.

$$\sum_{i=0}^{n} P(X) = V^2 \sum_{i=0}^{n} N^i + V^2 \sum_{i=1}^{n} N^i + V^2 \sum_{i=2}^{n} N^i + \ldots. + V^2 N^n = 1$$

Write: $\sum_{i=0}^{n} P(X) = \lim_{n \to \infty} V^2 \sum_{i=0}^{n} (i+1) N^i = 1$ (9)

$V^2 \sum_{i=0}^{n} (i+1)N^i = V^2 \sum_{i=0}^{n} i(N^i) + V^2 \sum_{i=0}^{n} N^i$ (10)

$\lim_{n\to\infty} V^2 \sum_{i=0}^{n} N^i = V$, y que: $(1-V) = N$

$V^2 \sum_{i=0}^{n} i(N^i) = V^2[0N^0 + 1N^1 + 2N^2 + 3N^3 + 4N^4 + \cdots (n-1)N^{n-1} + nN^n]$

$\boldsymbol{W} = \sum_{i=0}^{n} iN^i = 0N^0 + 1N^1 + 2N^2 + 3N^3 + 4N^4 + \cdots (n-1)N^{n-1} + nN^n$ (11)

We multiply (11) by N

$NW = 0N^1N^0 + 1N^1N^1 + 2N^1N^2 + \cdots (n-1)N^1N^{n-1} + nN^1N^n$

$= 0 + N^2 + 2N^3 + 3N^4 + 4N^5 + \cdots (n-1)N^n + nN^{n+1}$(12)

(11) –(12)

$W = 0N^0 + 1N^1 + 2N^2 + 3N^3 + 4N^4 + 5N^5 + \ldots + (n-1)N^{n-1} + \mathrm{n}N^n$

$-\mathrm{NW} = -(0 + N^2 + 2N^3 + 3N^4 + 4N^5 + \ldots + (n-2)N^{n-1} + (n-1)N^n + nN^{n+1})$

$W - \mathrm{NW} = N^1 + N^2 + N^3 + N^4 + N^5 + \cdots N^{n-1} + N^n - nN^{n+1}$

$\therefore W - \mathrm{NW} = N(1 + N^1 + N^2 + N^3 + N^4 + N^5 + \cdots N^{n-1} - nN^n)$

Write:

$1 + N^1 + N^2 + N^3 + N^4 + N^5 + \cdots N^{n-1} + N^n - (n+1)N^n = \left(\sum_{\mathbf{i}=0}^{\mathbf{n}} N^{\mathbf{i}}\right) - (n+1)N^n$

$= \sum_{\mathbf{i}=0}^{\mathbf{n}} \mathbf{N}^{\mathbf{i}} = \frac{1}{1-N}$

$W - \mathrm{NW} = \mathrm{N}\{(\frac{1}{1-N}) - (\mathrm{n}+1)\mathrm{N}^{\mathrm{n}})$

Sí, $\mathrm{n} \to \infty$: $\lim_{n\to\infty} \mathrm{N}^n = 0 \therefore (\mathrm{n}+1)\mathrm{N}^{\mathrm{n}} = 0$

$W - \mathrm{NW} = W(1-N) = \frac{\mathrm{N}}{1-N}$

$\Rightarrow W = \frac{\mathrm{N}}{(1-N)^2} \therefore W = \sum_{i=0}^{n} iN^i = \frac{\mathrm{N}}{(1-N)^2}$

we substitute (10)

$\lim_{n\to\infty}\left\{V^2 \sum_{i=0}^{n} \mathrm{i}\, N^{\mathrm{i}} + V^2 \sum_{i=0}^{n} N^{\mathrm{i}}\right\} =$

$= V^2\left[\frac{N}{(1-N)^2} + \frac{1}{1-N}\right] = (1-N)^2\left[\frac{N+1-N}{(1-N)^2}\right] = \mathbf{1}$(13)

For any value of V

$\lim_{n\to\infty}\{V^2 \sum_{i=0}^{n} i(N^i) + V^2 \sum_{i0}^{n} (N^i)\} = \boldsymbol{N} + \boldsymbol{V} = \mathbf{1}$

Generalizing, given that a probability function must satisfy:

$1. f(x) \geq 0,\quad 2\ \sum_{i=0}^{n} f(.\lim_{n\to\infty} x_i) = 1$

Previously we got:

$S = \sum_{i=0}^{n} N^i = \frac{1-N^n}{1-N} \Rightarrow \lim_{n\to\infty} S = \frac{1}{1-N},\ $ Sí $|N| < 1$

For this case we know that:

$0 < N < 1,\ V = 1 - N,$ Lo que cumple $|N| < 1$

Expected value:

$E(X) = \sum_{x=0}^{n} xpq^x = (0)pq^0 + 1pq + .. + npq^{n+1}$

$= \sum_{x=1}^{n} xpq^x = \sum_{x=1}^{n} xp(1-p)^x$

$= p\sum_{x=1}^{n} x(1-p)^{x-1+1} = p\sum_{x=1}^{n} x(1-p)^{x-1}(1-p)$

$= p(1-p)\sum_{x=1}^{n} x(1-p)^{x-1}$

Derivative: $\frac{d(1-p)^x}{dp} = (-1)x(1-p)^{x-1}$

$\therefore x(1-p)^{x-1} = -\frac{d}{dp}(1-p)^x$

$[-p(1-p)]\sum_{x=0}^{n} \frac{d}{dp}(1-p)^x$

$= p(1-p)\sum_{x=0}^{n} x(1-p)^{x-1}$

From the series: $\sum_{i=0}^{\infty} k^i$

$\sum_{i=0}^{\infty} k^i = \frac{1}{1-k}$ for: $-1 < k < 1$

$\sum_{i=0}^{n} (1-p)^i = \frac{1}{1-(1-p)} - 1 < (1-p) < 1$

$\frac{1}{p} = (1-p)^0 + \sum_{i=1}^{n} (1-p)^i$

$\frac{1}{p} - 1 = \frac{1-p}{p} = \sum_{i=1}^{n} (1-p)^i$

$\sum_{i=0}^{n} (1-p)^i = 1 + \sum_{i=1}^{n} (1-p)^i$

$\frac{1}{p} = 1 + \sum_{i=1}^{n} (1-p)^i \Rightarrow \sum_{i=1}^{n} (1-p)^i = \frac{1}{p} - 1$

$E(X) = -p(1-p)\frac{d}{dp}(\frac{1-p}{p})$

$= -p(1-p)\frac{p(-1)-(1-p)1}{p^2} = -p(1-p)(\frac{-1}{p^2})$

$\boldsymbol{E(X)} = \frac{p(1-p)}{p^2} = \frac{1-p}{p}$

$\boldsymbol{E(X) = \frac{q}{p}}$

To obtain Var(X), we resort to:

$E[X(X-1)] = \sum_{x=0}^{n} [X(X-1)]f(X)$

$= \sum_{x=0}^{n} [X(X-1)]pq^x$

$0(0-1)pq^0 + 1(1-1)pq + \sum_{x=2}^{n} [X(X-1)]pq^x$

$= 0 + 0 + \sum_{x=2}^{n} [X(X-1)]pq^x$

$p(1-p)^2 \sum_{x=2}^{n} [X(X-1)](1-p)^{x-2}$

Derivative: $\frac{d^2}{dp^2}(1-p)^x = x(x-1)(1-p)^{x-2}$

$\therefore E[X(X-1)] = p(1-p)^2 \frac{d^2}{dp^2} \sum_{x=2}^{n} (1-p)^x$

$E[X(X-1)] = p(1-p)^2 \sum_{x=2}^{n} [X(X-1)](1-p)^{x-2}$

$= p(1-p)^2 \frac{d^2}{dp^2} \sum_{x=2}^{n} (1-p)^x$

$\sum_{i=0}^{\infty} k^i$

$\sum_{i=0}^{\infty} k^i = \frac{1}{1-k} \quad -1 < k < 1$

for: $-1 < (1-p) < 1$

$\sum_{i=0}^{n} (1-p)^i = \frac{1}{1-(1-p)} = \frac{1}{p}$

$\frac{1}{p} = (1-p)^0 + (1-p)^1 + \sum_{i=2}^{n} (1-p)^i$

$\sum_{i=2}^{n} (1-p)^i = \frac{1}{p} - 2 + p = \frac{1-2p+p^2}{p}$

$\sum_{x=2}^{n} (1-p)^x$

$E[X(X-1)] = p(1-p)^2 \frac{d^2}{dp^2}(\frac{1-2p+p^2}{p})$

$= p(1-p)^2 \frac{d}{dp}(\frac{p(2p-2)-(1-2p+p^2)}{p^2})$

$= p(1-p)^2 \frac{d}{dp}(\frac{2p^2-2p-1+2p-p^2}{p^2})$

$= p(1-p)^2 \frac{d}{dp}(\frac{p^2-1}{p^2})$

$= p(1-p)^2 \frac{p^2(2p)-(p^2-1)2p}{p^4}$

$= p(1-p)^2 \frac{2(p^3-p^3+p)}{p^4} = \frac{2(1-p)^2}{p^2}$

$E(X) = \frac{q}{p} = \frac{1-p}{p}$

$\therefore E[X^2] - E[X] = \frac{2(1-p)^2}{p^2}$

$\Rightarrow E[X^2] = \frac{2(1-p)^2}{p^2} + \frac{1-p}{p}$

$= \frac{2(1-2p+p^2)+p(1-p)}{p^2}$

$= \frac{2-4p+2p^2+p-p^2}{p^2} = \frac{p^2-3p+2}{p^2}$

$E[X^2] = \frac{p^2-3p+2}{p^2}$

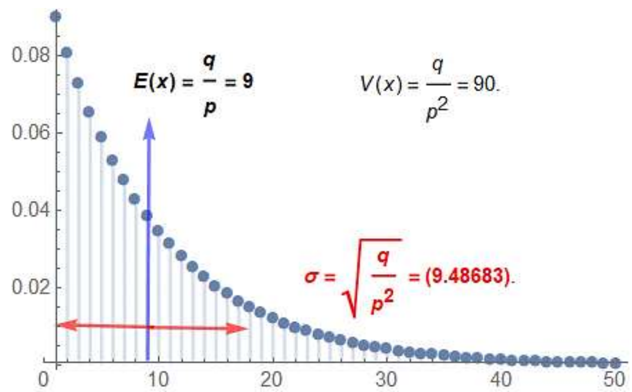
$\mathbf{Var}\boldsymbol{(X)} = E(X^2) - [E(X)]^2$

$= \frac{p^2-3p+2}{p^2} - \frac{(1-p)^2}{p^2}$

$= \frac{p^2-3p+2-1+2p-p^2)}{p^2} = \boldsymbol{\frac{1-p}{p^2}}$

$\mathbf{Var}\boldsymbol{(X) = \frac{1-p}{p^2}}$

DiscretePlot[Evaluate@Table[PDF[Geometric Distribution[p],n],{p,{0.1}}],{n,50},PlotMarkers->Automatic,PlotRange->All]



Graph2

Conclusions

In this work, concepts and proofs have been presented using clear and detailed language sufficient for a higher level student of mathematics. It is possible that these didactic strategies do not allow covering teaching cases in other higher level areas where there are subjects of probability and statistics in the school curriculum. These areas can be Engineering, Biology, Ecology and Sociology. In order for the student to understand the meanings of the fundamental concepts of probability and its procedures, the use of algebraic artifices supported by differential and integral calculus is encouraged.

Other important aspects that were taken into account are:

a) That the student acquire mastery and understanding of the basic concepts of Axiomatic Probability.

b) That students develop the autonomous character to build and develop logical arguments under a demonstration.

c) It is hoped that students can express themselves correctly using the formal language of Mathematics based on already acquired knowledge.

Suggestions (future work)

To further enrich this didactic approach, it is suggested that the demonstrations be developed by means of the moment generating functions as another way to obtain the expressions of expected value and variance of the main probability distributions.

It is also suggested to use mathematical software such as R, Matlab, Wolfram Mathematica to simulate the behavior of the main probability distributions with examples of real applications. It is suggested to put this proposal into practice to groups of higher level students that allow them the opportunity to understand the conceptual foundations to develop mathematical demonstrations.

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