Level of competence on the concept of basis of vector space using SOLO taxonomy

Nivel de competencia del concepto de base de un espacio vectorial usando la taxonomía SOLO

VALENZUELA-GONZÁLEZ, Verónica†*, HERNÁNDEZ-QUINTANA, Andrés and CAMACHO-RÍOS, Alberto

Tecnológico Nacional de México campus Chihuahua II, Mexico.

ID 1st Author*: Verónica, Valenzuela-González* / **ORC ID:** 0000-0002-4363-4930, **Researcher ID Thomson:** AAN-7720-2021. **CVU CONACYT ID:** 1134175

ID 1st Co-author: *Andrés, Hernández-Quintana* / **ORC ID:** 0000-0002-3486-4400

ID 2nd Co-author: *Alberto, Camacho-Ríos* / **ORC ID:** 0000-0002-0685-4723, **Researcher ID Thomson:** C-6849-2017, **CVU CONACYT ID:** IT18C618

DOI: 10.35429/JMQM.2021.8.5.10.16 Received January 15 2021; Accepted June 30, 2020

Abstract

The purpose of this research paper is to determinate the level of competence on the concept of basis of a vector space on college level students who finished the Linear Algebra course, providing complementary information to the various studies from a different perspective through the SOLO taxonomy (Structure of the Observed Learning Outcome). An instrument was designed to address the concept of basis from different perspectives and difficulty degrees; it was applied through individual video-recorded interviews. Their answers were analyzed, and it was found that they average level 3 (multistructural) of the taxonomy: the students know the basis definition from an algorithmic or methodological perspective, and they can reproduce some procedures, but they are unable to understand the basis concept.

Vector space, Basis, SOLO taxonomy

Resumen

El propósito de esta investigación es determinar el nivel de competencia sobre el concepto de base de un espacio vectorial en estudiantes de nivel superior que concluyeron el curso de Álgebra Lineal, proporcionando información complementaria a diversos estudios desde una perspectiva diferente a través de la taxonomía SOLO (Estructura del Resultado Observado del Aprendizaje por sus siglas en inglés). Se diseñó un instrumento abordando el concepto de base desde diferentes perspectivas y grados de dificultad; aplicando entrevistas individuales videograbadas. Analizando sus respuestas, se encontró que en promedio alcanzan el nivel 3 (multiestructural) de la taxonomía: los estudiantes conocen la definición de base desde una perspectiva algorítmica o metodológica, pueden reproducir algunos procedimientos, pero son incapaces de comprender el concepto de base.

Espacio vectorial, Base, taxonomía SOLO

Citation: VALENZUELA-GONZÁLEZ, Verónica, HERNÁNDEZ-QUINTANA, Andrés and CAMACHO-RÍOS, Alberto. Level of competence on the concept of basis of vector space using SOLO taxonomy. Journal-Mathematical and Quantitative Methods. 2021. 5-8:10-16.

* Correspondence to Author (Email: veronica.vg@chihuahua2.tecnm.mx).
† Researcher contributing as first author.

Introduction

In the Linear Algebra course taught in the engineering careers of the Tecnológico Nacional de México (TecNM), it has been detected that student have greater difficulty in understanding and assimilating the concepts related to the subject of vector spaces. There are multiple investigations that address this problem and they refer to the abstract nature and complexity of these concepts (Guzman and Zambrano 2015; Vera and Miranda 2014; Parraguez and Vera-Soria 2020).

The base concept constitutes a fundamental element of the structure of a vector space and is fundamentally related to other concepts (Ku, Trigueros and Oktaç, 2008). For this reason, it is useful to evaluate the assimilation of this concept to obtain a sample of the understanding that students have of the subject of vector spaces.

The Linear Algebra course is found within several university-level study programs (TecNM), especially in those engineering programs where the concepts and methods are retaken for their application in subsequent subjects.

This study attempts to answer the following research question: What level of competence do students have about the concept of the basis of a vector space at the end of the Linear Algebra course?

To answer this question, an instrument was designed that measures the level of competence of the basic concept, using the SOLO taxonomy whose levels show the degree of understanding that students have in a given topic, the instrument is found in Annex 1. The instrument was applied through a video-recorded interview and the responses of each student were analyzed.

Background

In recent decades, interest has arisen in the study of Linear Algebra, specifically in the understanding of vector spaces, Dubinsky (2001) mentions the problems faced by students experiencing “confusion and disorientation” when analyzing subspace topics, generator set, linear independence, etc.

Some investigations related to the concept of base are those of Ku (2008) which is oriented in the study of the understanding of the concept of base of a vector space. Oktac (2010) delves into how university students learn linear algebra considering the construction of the concept of vector space, linear transformations and base. Parraguez (2013) explains the role of the body in the construction of the vector space concept. Vera (2018) makes an inquiry about the process of its construction, through the assessment of the different ways of perceiving the basic meaning. The aforementioned investigations have been carried out from the APOE theory (Action, Process, Object and Scheme) developed by Dubinsky.

Martín, *et al*., (2014) reviewed several articles that show interest in the teaching-learning topics of Linear Algebra specifically where the topics of Vector Spaces are involved. Madrid, Cribeiro and Sánchez (2016), after investigating and carrying out an analysis of the difficulties in learning the subjects of vector spaces, developed worksheets that led the student to develop the necessary skills on the concept of vector space.

Theoretical framework

Concepts

There are some basic knowledge on which most of the concepts related to vector spaces are based, the most relevant are those of Base, Dependency and Linear Independence and generated space that, according to Grossman and Flores (2012) are defined as follows:

Base:

A finite set of vectors {v1, v2, …, vn} is a basis for a vector space V if

i) {v1, v2, …, vn} is linearly independent.

ii) {v1, v2, …, vn} generates V

Linear dependence and independence:

Let v1, v2, …, vn, n vectors in a vector space V. Then the vectors are said to be linearly dependent if there are n scalars c1, c2, …, cn not all zero such that:

$$c_1v_1 + c_2v_2 + \ldots + c_nv_n = 0$$

ISSN-On line: 2531-2979

If the vectors are not linearly dependent, they are said to be linearly independent.

Generated space

Let v1, v2, …, vk be vectors of a vector space V. The space generated by {v1, v2, …, vk} is the set of linear combinations v1, v2, …, vk. That is, gen {v1, v2, …, vn} = {v: v = a1v1 + a2v2 +… + akvk}

Where a1, a2, ..., ak are arbitrary scalars.

SOLO taxonomy

The structure of the observed learning outcome, known as the SOLO taxonomy (Structure of the Observed Learning Outcome): "It provides a systematic way of describing how the complexity of the student's learning increases as he masters academic tasks" (Biggs, 2006).

The SOLO taxonomy has the following levels:

1.- Prestructural: they do not give proof of relevant learning. The student does not understand the topic.

2.- Unistructural: it only fulfills a part of the task, they stay in the terminology, they are well oriented, but only in one aspect. At this level the student identifies or performs a simple procedure.

3.- Multistructural: performs a series of tasks, which can be enumerate, describe, make a list, combine and are capable of developing algorithms.

4.- Rational: not only do they have a set of data and details, they address a point and give meaning to the topic, it has a relevant meaning and leads to understanding. The student compares, explains causes, analyzes, relates and applies knowledge

5.- Extended abstract: it is conceptualized at a higher level of abstraction and applied to new, broader fields. The student theorizes, generalizes, formulates hypotheses and reflects.

This theory allows evaluating learning outcomes, giving information on the level the student is at and giving the opportunity to search for specific strategies to lead the student to reach a new level of knowledge.

Methodology

The participants in the study were 18 students from two different groups of the second semester of the Computer Systems Engineering career who finished the course of Linear Algebra at the TecNM, Chihuahua II campus, who were invited randomly.

An individual interview was conducted through which the items were provided one by one, which were designed as follows:

- In the first five questions they were asked to determine if a set of vectors in R2 or R3 is a basis for the indicated vector space.

- In the following three questions it was suggested to determine if the set of vectors is a basis for the vector space indicated, using second-degree polynomials and 2x2 matrices.

- In the last three questions, they were asked to determine a base for the entire set of points of a plane in three dimensions, of a line, also in three dimensions, and for the set of 3x3 diagonal matrices.

The complete reagents can be seen in Annex 1.

In this part, the strategy consisted in that, if the professor who carried out the interview detected arithmetic errors in the students' procedures, he guided them to correct their operations in order to reach a conclusion. The interviewer questioned the student about his conclusions in order to inquire about the level of understanding of the basic concept.
The interview was videotaped to later establish the analysis of the responses provided by the students, which they reached through operations, or verbally expressing their justification regarding their conclusions.

Therefore, the study is descriptive-interpretive, since the recordings of the interviews were analyzed to determine the level of competence that each of the participants had.

Results

Students were located at each level as shown below.

Level 1. Prestructural

No student was found at this level, as all showed at least a basic notion of the basic concept.

Level 2. Unistructural

Three of the eighteen students were at this level, they mastered a single task and those items that did not correspond to the latter, which they could not solve, were left unanswered. According to the procedures shown, these students were able to obtain the determinant, or, form a matrix and match to zeros to determine linear independence through definition, although some used row reduction to determine whether it corresponded to a generator set.

Thus, for example, one of the students, at the unistructural level, proposed a procedure to verify if the set of vectors generated the vector space in all the exercises, without mentioning linear independence. However, there were cases in which, even with the procedure, they did not conclude correctly, this is the case of the reagent presented below:

Determine if the given set of vectors is a basis for the indicated vector space:

$$\left\{\begin{bmatrix}-3\\1\\2\end{bmatrix},\begin{bmatrix}4\\0\\-1\end{bmatrix}\right\} \text{In } R^3 \qquad (1)$$

In the following figure you can see the procedure performed by the student.

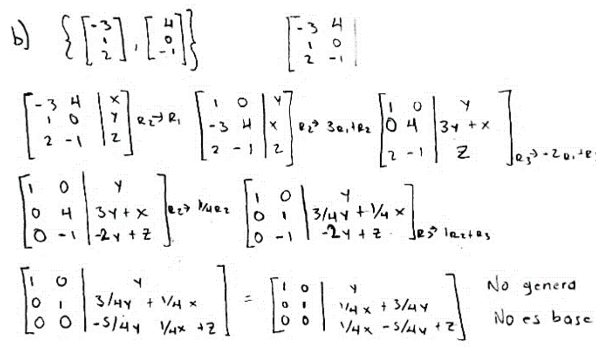

Figure 1 Photograph of the procedure of exercise I.b carried out by a student who reached level 2

Even when the student carried out the procedure that is used daily to determine the space generated by vectors, he could not interpret it, he concluded: “It is not a base since I have zero equal to something of x, y, z. It is not consistent, a set of zeros cannot be equal to x, y, z".

Level 3. Multistructural

Eight of the eighteen students performed various procedural tasks, applied the definition using row reduction, and concluded whether the vectors in R2 or R3 were linearly independent. They also carried out procedures to determine if this was a generator set and to conclude, also, if it corresponded to a basis, or not, of a vector space. They were not able to extrapolate the concepts of base, linear independence, and generating set to vector spaces composed of polynomials or matrices. Therefore, they were placed at level 3, because they performed algorithmic procedures without achieving a broader understanding of the underlying concept, as exemplified below.

One of the students who was able to successfully solve the questions, in which he wondered if a certain number of vectors were the basis of R2 or R3, had problems to raise those questions of matrices or polynomials:

Determine if the given set of vectors is a basis for the indicated vector space.

$$\left\{\begin{bmatrix}3 & -2\\0 & 0\end{bmatrix},\begin{bmatrix}2 & 0\\-1 & 0\end{bmatrix}\right\} \text{ for } M_{22} \qquad (2)$$

In the following figure you can see the procedure of one of the students.

ISSN-On line: 2531-2979

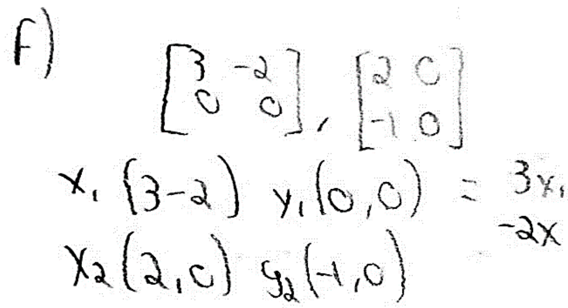

Figure 2 Photograph of the procedure of exercise I.f carried out by a student who reached level 3

The use of elements x and y that belong to the vector space R2 is appreciated, without establishing a relationship of the concepts of linear independence or space generated with the given matrices.

Level 4. Rational

Five of eighteen students were at level 4, in which they performed analysis through knowledge of the theorems or performed mathematical procedures justifying each of their operations to conclude correctly. The latter mastered the operations, with which they managed to determine the linear independence either by the definition or by the theorem, as well as the generator set, in addition, they easily determined if the vectors, polynomials or matrices shown are the basis of the space vector. They were able to solve some of the questions in which they were asked about a basis for a vector space without being certain about their answer, showing little understanding of the concept of dimension.

In the case of a student who was able to answer the questions satisfactorily, it was necessary to determine if the set of vectors is the basis of the vector space, he showed difficulties in proposing a basis for a vector space, as shown in the following example:

Determine a basis for the set of vectors in the plane $3x - 2y + z = 0$ in R^3.

The student proposed as a basis the vectors that can be seen in the following figure.

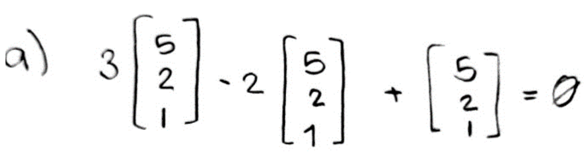

Figure 3 Photograph of the procedure of exercise II.a carried out by a student who reached level 4

He answered verbally "It comes to my mind to give values to x, y, z (pointing to the elements of the vector) so that it gives zero"

As can be seen, the student substituted a random vector in x, y, z, since they should be components of the vector and not vectors.

Level 5. Broad Abstract

Two of the eighteen students were at this level, they showed mastery of row operations to determine linear independence and the generator set, as well as the use of theorems to conclude whether it is base or not, solving those that include polynomial and matrices. In the items in which they had to propose a basis for a given vector space, they were able to analyze and determine the dimension, which corresponds to the number of vectors to be used. Thus demonstrating the domain and understanding of the concept of base of a vector space.

One of the students, in this last level, when proposing the reagent, explained the following:

Determine a basis for the set of 3 X 3 diagonal matrices.

Established the following dialogue with the interviewer

Student: The set of diagonal matrices?

Interviewer: What are the diagonal matrices?

Student: those with zeros above and below (pointing to a diagonal).

Interviewer: What would be a basis for those matrices?

Student: For all of them?

Interviewer: There they specify which ones, of 3 X 3

Student: Yes, it would be 3 linearly independent matrices.

The following figure 4 shows the student's response.

ISSN-On line: 2531-2979

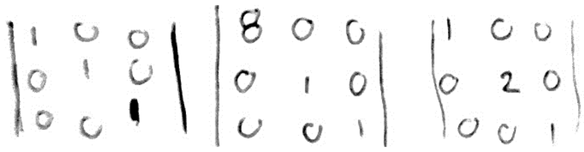

Figure 4 Photograph of the procedure of exercise II.b carried out by a student who reached level 5

He easily intuited that he needed three linearly independent diagonal matrices, applying the concept of base and dimension to a type of problem that is not seen in class.

Conclusion

The results obtained showed that the average level of competence presented by the students is multistructural in nature, arithmetically averaging 3.33, which means that they know the basic definition from an algorithmic or methodological perspective, managing to reproduce some procedures without having an understanding of the concepts related to the basis of a vector space.

Knowing that the base concept of a vector space is abstract, it is convenient for students to develop activities that help better understanding, as proposed by Madrid, Cribero and Sanchez (2016).

Annex 1 Instrument

I.- Determine if the given set of vectors is a basis for the indicated vector space.

$a\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}-1\\4\end{bmatrix},\begin{bmatrix}3\\1\end{bmatrix}\right\} enR^2$

$b\left\{\begin{bmatrix}-3\\1\\2\end{bmatrix},\begin{bmatrix}4\\0\\-1\end{bmatrix}\right\} enR^2$

$c\left\{\begin{bmatrix}3\\-1\\2\end{bmatrix},\begin{bmatrix}4\\1\\5\end{bmatrix},\begin{bmatrix}-2\\-1\\-3\end{bmatrix}\right\} enR^3$

$d\left\{\begin{bmatrix}-1\\0\\1\end{bmatrix},\begin{bmatrix}0\\-1\\1\end{bmatrix},\begin{bmatrix}1\\-1\\0\end{bmatrix}\right\} enR^3$

$e\left\{\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix}\right\} enR^2$

$f\left\{\begin{bmatrix}3 & -2\\0 & 0\end{bmatrix},\begin{bmatrix}2 & 0\\-1 & 0\end{bmatrix}\right\} paraM_{22}$

$g\{2t^2-t, t+4, t^2+1\} paraP_2$

$h\{x^2-4, 3x^2, 5\} paraP_2$

II.- Determine a basis for:

a) In R^3 for the set of vectors in the plane 3x-2y + z = 0

b) In R^3 for the set of vectors on the line x = 4t, y = -2t, z = -t

c) In M_{33} for the set of diagonal matrices.

References

Grossman, S. I., & Flores, J. J. (2012). *Álgebra Lineal* (7th ed.). Mc Graw Hill.

Guzman, J., & Zambrano, J. (2015). Base de un espacio vectorial de Rn y tecnología. *XIV Conferencia Interamericana de Educación Matemática*.

Parraguez, M., & Vera-Soria, G. (2020). Los modos de pensamiento sintético y analítico en la comprensión de base en el espacio vectorial R2: Un estudio de casos en un contexto universitario. *Revista Paradigma*, *41*, 600–365.
Vera, M. G., & Miranda, E. (2014). El aprendizaje de espacios vectoriales en ambientes computacionales. *Acta Latinoamericana de Matemática Educativa*, *27*, 2195–2203.

Martín, Á., Pérez, O., Blanco, R., & Casas, L. (2014). Los espacios vectoriales, como estructuras algebraicas, en el proceso de enseñanza aprendizaje del álgebra lineal: Una propuesta de investigación. *Acta Latinoamericana de Matemática Educativa*, *27*, 1073–1082. http://funes.uniandes.edu.co/5666/1/MartinLoseспaciosALME2014.pdf

Vera, M. G., Parraguez, M., Paredes, I., & Garcia, D. (2018). Los modos de pensamiento en el proceso de comprensión del concepto de base del espacio vectorial R2 y R3. *Acta Latinoamericana de Matemática Educativa*, *31*(2), 1217–1225.

Madrid, H., Cribeiro, J. M., & Sánchez, M. L. (2016). Espacios vectoriales desde la realidad a la abstracción. *El Cálculo y Su Enseñanza*, *7*, 1–8.

ISSN-On line: 2531-2979

Dubinsky, E. (2001). Some Thoughts on a first course in Linear Algebra at the college level. *MAA NOTES*, *620*, 85–106.

Kú, D., Trigueros, M., & Oktaç, A. (2008). Comprensión del concepto de base de un espacio vectorial desde el punto de vista de la teoría APOE. *Educación Matemática*, *20*(2), 65–89. http://www.scielo.org.mx/scielo.php?script=sci_arttext&pid=S1665-58262008000200004

Biggs, J. (2006). *Calidad Del Aprendizaje Universitaria* (2nd ed.). Narcea, SA de ediciones.

Parraguez González, M. (2013). El rol del cuerpo en la construcción del concepto Espacio Vectorial. *Educación Matemática*, *25*(1), 133–154.

Oktaç, A., & Trigueros, M. (2010). ¿Cómo se aprenden los conceptos de álgebra lineal? *Revista Latinoamericana de Investigación En Matemática Educativa, RELIME*, *13*(4), 373–385.