# Mathematical Modeling of a Variable Gain PID Controller, Dependent of the set point in the output of a Conventional PID

# Modelado Matemático de un Controlador PID de Ganancia Variable dependiente del Punto de Consigna en la Salida de un PID Convencional

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**Abstract**

A PID control structure was modified proposed by K. Ogata. It is proposed a structure Nonlinear PID control, implemented using analogical electronics. A mathematical model was obtained using conventional methods. It was proved the validity of the model obtained comparing against the response of the model proposed by K. Ogata. It was concluded that the proposed structure is stable.

**Modeling, Nonlinear control systems, PID**

**Resumen**

Se modificó la estructura de control PID mostrada en la literatura por K. Ogata. Se hace la propuesta de una estructura de control PID de ganancia variable, implementada mediante electrónica analógica. Se obtuvo un modelo matemático utilizando métodos convencionales. Se comprobó la validez del modelo obtenido comparando contra la respuesta del modelo planteado por K. Ogata. Se concluyó que la estructura propuesta es estable.

**Modelado, Sistemas de Control no lineales, PID**

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## Introduction

This paper shows the mathematical model obtained for a variable gain PID control structure implemented by operational amplifiers. To achieve non-linear behavior in the controller, a N-channel JFET transistor was added to the input of the operational amplifier that executes a conventional PID control algorithm. In this case, it is required to control the internal temperature of an electric furnace that heats by means of 2 electrical resistors connected in series.

This system has a high thermal time constant. The oven operates at 127 Volts, and consumes 6 Amps. We have been working with this type of modifications to conventional control algorithms with the aim of achieving a faster and more accurate control of temperature, for applications of heat treatment of metallographic samples.

The modified PID controller considered in this paper is designed to increase its gain as the setpoint is adjusted to a larger value. The proposed Nonlinear PID Control has been experimentally tested successfully. The advantages of a PID control are preserved, with the novelty that the gain of the controller will be greater when the reference point is increased, reinforcing the control voltage level (Vcon), making the reference point of the variable reach more quickly than with the conventional version, and staying at the desired value.

## Methodology

The PID control structure implemented by operational amplifiers shown in Figure 1 is the one originally proposed by K. Ogata in [1].

The author explains that the second AO works as a sign invertor and as a gain adjuster.

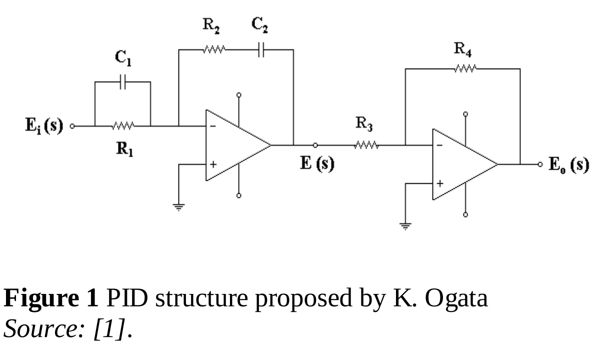


**Figure 1** PID structure proposed by K. Ogata
*Source: [1].*

Taking the circuit of Figure 1 as a reference, the transfer function that results from its analysis is equation (1).

$$\frac{E_o(s)}{E_i(s)}=\frac{R_4(R_1C_1+R_2C_2)}{R_3R_1C_2}\left[1+\frac{1}{(R_1C_1+R_2C_2)s}+\frac{R_1C_1R_2C_2}{R_1C_1+R_2C_2}s\right] \quad (1)$$

Equation (2) is the scheme to which this structure corresponds, O'Dywer calls it: "ideal PID controller" [2]. When the controller is expressed with the canonical transfer function of equation (2), Kp is called proportional gain, Ti integral time and Td derivative time [1].

$$\frac{E_o(s)}{E_i(s)}=K_p\left(1+\frac{T_i}{s}+T_d s\right) \quad (2)$$

From equation (1), we obtain that the proportional gain Kp, the integral time Ti and the derivative time Td are:

$$K_p=\frac{R_4(R_1C_1+R_2C_2)}{R_3R_1C_2} \quad (3)$$

$$T_i=\frac{1}{R_1C_1+R_2C_2} \quad (4)$$

$$T_d=\frac{R_1C_1R_2C_2}{R_1C_1+R_2C_2} \quad (5)$$

The ideal PID control scheme can also be expressed through the canonical transfer function shown in (6), in this case, Kp is called proportional gain, Ki integral gain and Kd derivative gain [1].

$$\frac{E_o(s)}{E_i(s)}=K_p+\frac{K_i}{s}+K_d s \quad (6)$$

For this controller:

$$K_p=\frac{R_4(R_1C_1+R_2C_2)}{R_3R_1C_2} \quad (7)$$

$$K_i=\frac{R_4}{R_3R_1C_2} \quad (8)$$

$$K_d=\frac{R_4R_2C_1}{R_3} \quad (9)$$

To obtain the gains of the structure originally proposed by K. Ogata, the commercial values determined in a previous work are used [3]. The circuit is shown in Figure 2. The gains result:

$K_p=56.21$ según (7)

$K_i = 21276.59$ según (8)

$K_d = 5.6x10^{-4}$ según (9)

Substituting these values in (6) is written:

$$\frac{E_o(s)}{E_i(s)} = 56.21 + \frac{21276.59}{s} + 0.00056 \cdot s \quad (10)$$

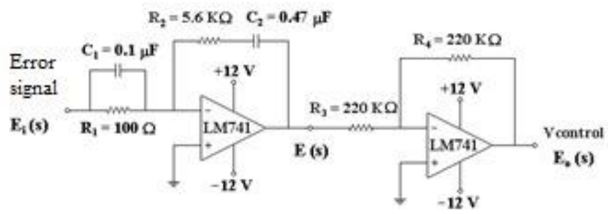


**Figure 2** Values determined after a fine tuning in the site for the PID structure proposed by K. Ogata.
*Source: [3].*

**Development**

The Normal PID circuit proposed by Ogata, is modified to make PID of Variable Gain depending on the reference point, with the feedback loop adjusted in the output of the conventional PID controller. For this, a JFET Transistor is added as shown in Figure 3.

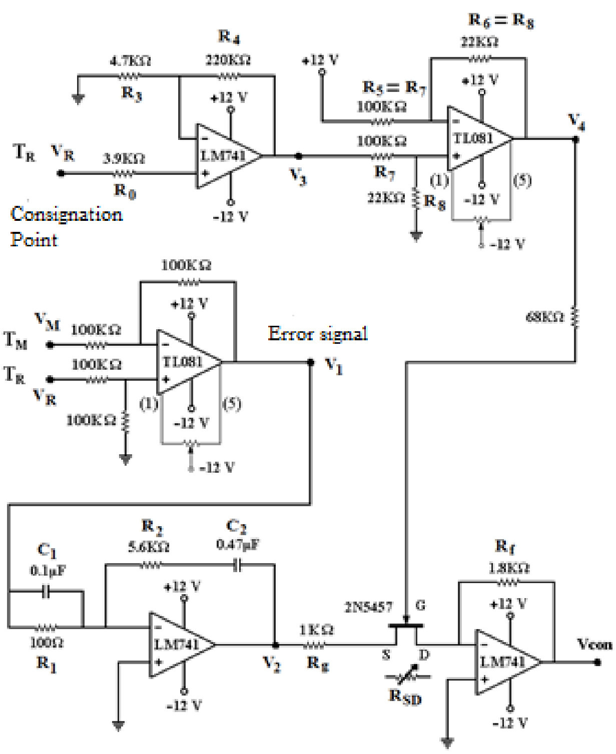


**Figure 3** Nonlinear PID structure obtained by adding a JFET Transistor
*Source: Own Elaboration*

The idea is to adapt the electronic signal that represents the set point to excite the Gate terminal of the JFET transistor and make it work as a variable Source-Drain (RSD) resistor. The value of RSD will depend on the magnitude to which the desired reference point is adjusted and act to adjust the gain of the PID controller and make its behavior non-linear, similar to that presented in [4].

This voltage is first amplified in the AO configured as a non-inverting amplifier by multiplying it by 46.8, then in the AO configured as a subtracter, two operations are performed simultaneously, first subtracting from -12 volts and then divided by 4.54. In this way, depending on the value of the reference voltage, the output voltage of the AO configured as a subtracter will have a value between 0 to 0.7 volts positive with which the JFET transistor will behave as a variable resistance, approximately from 100Ω to 1000Ω.

Continuing with the explanation of Figure 3, the voltage representing the error signal is entered into the Operational Amplifier where the normal PID control was configured; Then in the sign inversion stage, the resistance Rg is connected in series to the Source terminal of the JFET transistor, whereby the internal resistance RSD of 2N5457, is added in series with the resistance Rg. The above, together with the value of Rf is what will cause the gain of the controller to vary depending on the magnitude of the selected reference value.

The commercial values shown in Figure 3 were obtained experimentally, achieving a good performance in the required application.

The equations describing the non-linear PID control shown in Figure 3 are established from the impedance method described by K. Ogata in [1].

Defining:

$$V_1 = V_R - V_m \quad (11)$$

$$V_2 = -\frac{R_2}{R_1}\left[\frac{(R_1C_1S+1)(R_2C_2S+1)}{R_2C_2S}\right]V_1$$

Simplifying

$$V_2 = -\left(R_2C_1S + \frac{1}{R_1C_2S} + \frac{R_1C_1+R_2C_2}{R_1C_2}\right)V_1 \quad (12)$$

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Substituting $V_1$ in $V_2$ you have:

$$V_2 = -\left(R_2C_1S + \frac{1}{R_1C_2S} + \frac{R_1C_1+R_2C_2}{R_1C_2}\right)(V_R - V_m) \quad (13)$$

Analysis of the setpoint

$$V_3 = \left(1 + \frac{R_3}{R_4}\right)V_R \quad (14)$$

$$V_4 = \frac{R_6}{R_5}(V_3 - 12) \quad (15)$$

Substituting $V_3$ in $V_4$ you have:

$$V_4 = \frac{R_6}{R_5}\left[\left(1 + \frac{R_3}{R_4}\right)V_R - 12\right]$$

$V_G = V_4$ Por lo tanto

$$V_G = \frac{R_6}{R_5}\left[\left(1 + \frac{R_3}{R_4}\right)V_R - 12\right] \quad (16)$$

Analyzing the JFET

$$\frac{V_{control}}{V_2} = -\left[\frac{R_f}{R_g + R_{SD}}\right]$$

$$V_{control} = -\left[\frac{R_f}{R_g+R_{SD}}\right]V_2 \quad (17)$$

Substituting $V_2$ in $V_{control}$ is obtained:

$$V_{control} = -\frac{R_f}{(R_g + R_{SD})}\left[-\left(R_2C_1S + \frac{1}{R_1C_2S} + \frac{R_1C_1 + R_2C_2}{R_1C_2}\right)(V_R - V_m)\right](V_R - V_m)$$

Simplifying and due to the effect of the sign inverter results:

$$V_{control} = \frac{R_f}{(R_g+R_{SD})}\left[\left(R_2C_1S + \frac{1}{R_1C_2S} + \frac{R_1C_1+R_2C_2}{R_1C_2}\right)\right](V_R - V_m) \quad (18)$$

## Results

In order to verify the validity of the mathematical model, it was decided to compare the stability analyzes of the model studied by K. Ogata in [1] against the modification proposed in this work.

First the graphs for the model indicated by K. Ogata are obtained. To work with Matlab (10) is written:

$$\frac{E_o(s)}{E_i(s)} = \frac{0.00056 \cdot s^2 + 56.21 \cdot s + 21276.59}{s} \quad (19)$$

The function obtained from the model is improper, so it was decided to determine the stability by obtaining and analyzing the Bode diagrams.

In Figure 4, the Bode traces of (19) obtained with the help of MatLab are shown, their analysis indicates that the structure is stable, since the phase curve does not even cross the -180 degrees.

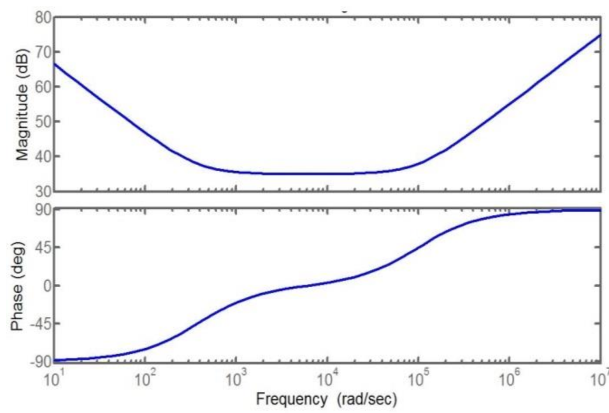


**Figure 4** Traces of Bode for the original structure of K. Ogata
*Source: Own Elaboration*

On the other hand, for the proposed variable gain PID structure, it is considered an intermediate moment of the control process. The above given that the RSD gain is variable. Table I shows a correspondence of the values that RSD acquires as the setpoint changes. Note that as the reference value increases, the value of RSD decreases, this will cause that at higher setpoints, the gain of the controller is greater. Measurements were made on site and then the formula was applied:

$$R_{SD} = \frac{V_{SD}}{I_{SD}} \quad (20)$$

| $T_R$ (V) | $R_{SD}$ (Ω) |
|---|---|
| 0.021 | 875 |
| 0.0405 | 785.7143 |
| 0.05 | 744.186 |
| 0.0758 | 645.3382 |
| 0.101 | 567.9443 |
| 0.126 | 500 |
| 0.1503 | 451.6129 |
| 0.1756 | 417.3228 |
| 0.2 | 406.25 |
| 0.2508 | 395.3488 |

**Table 1** RSD values in different reference points

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To obtain the transfer function it is enough to recognize that ($V_R$-$V_m$) is the voltage that represents the error signal, and therefore it is the voltage signal that in this case is considered as the input.

Now, if the reference point is set to 200 °; according to Table I, a value for RSD of 406.25 Ω can be assumed. Substituting this and the values shown in Figure 3, in (18), you can write:

$$\frac{V_{control}}{(V_R - V_m)} = 1.28\left(0.00056\, S + \frac{21{,}276.5957}{S} + 56.21\right)$$

Simplifying in MatLab you get:

$$\frac{V_{control}}{(V_R - V_m)} = \frac{0.00001491S^2 + 1.497S + 566.5}{S} \quad (21)$$

It should be noted that for reasons of space the coefficients of (21) have been rounded, however, in the calculations in Matlab if the complete quantities were considered.

In Figure 5 the Bode diagram for the case considered is shown. Analyzing the graph it is observed that the system is stable, since the phase curve does not even cross the axis of -180 degrees.

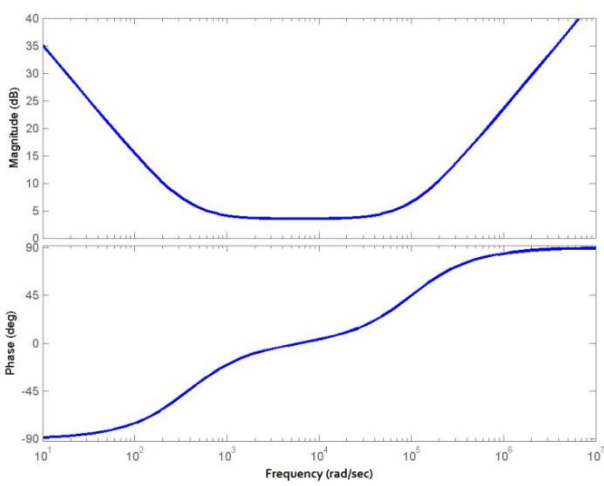


**Figure 5** Traces of Bode for the non-linear PID structure of (21)
*Source: Own Elaboration*

On the other hand, Figure 6 shows the results of a test case with the reference value set at 200 ° C. It includes the response curve that an ideal controller would have, which was obtained by performing an empirical modeling of the on-site control system.

This ideal model is adjusted to reach the set point at a time very close to that which would be in open loop, which would be very difficult to achieve without having a considerable overshoot, but it does provide an idea of how efficient the actual controllers are under study. Then, comparing the behavior of the conventional PID controller with the behavior of the proposed variable gain PID controller, it is verified that there is a slight improvement in the desired operation.

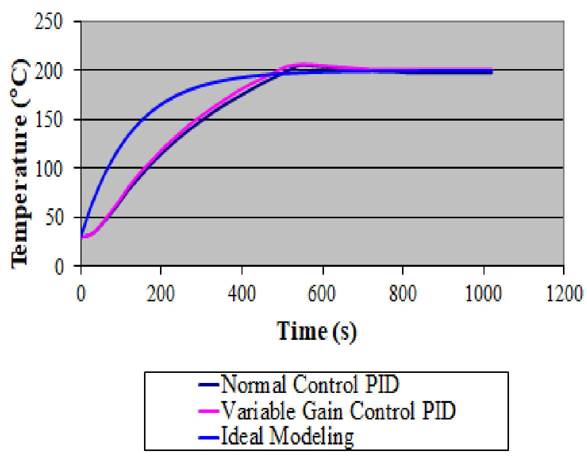


**Figure 6** Graph of results of real tests at 200 ° C
*Source: Own Elaboration*

## Conclusions

A mathematical model was obtained for a PID controller with gain adjustment implemented with Operational Amplifiers. The model provides an approximation of the real behavior of the proposed circuit and makes it possible to perform stability analysis, by which it was determined that the structure implemented is stable.

The model will serve as a basis for future work, where it is intended to use the structure to control SISO systems that have long delay times or that exhibit non-linear behavior.

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